Analysis of Decentralized Control Structures for Nonlinear Systems

Configuration of feedback loops between input and output variables of a given plant constitutes a subtask of control system synthesis. In the past, this subtask has been tackled through direct decomposition of either the process or the set of input-output variables. Linear interaction analysis has been used in both approaches to assess and minimize subsystem interactions. Since most chemical processes exhibit nonlinear behavior, it is evident that a measure is needed for assessment of interactions in the presence of system nonlinearities. In this paper we focus on input-output variable set decomposition and introduce the notion of nonlinear block relative gain (NBRG) as a nonlinear interaction measure. Both the statistic and dynamic versions of NBRG are discussed, and a computational procedure is presented for their evaluation. Direct simulations on a CSTR verify the interactions predicted by NBRG for different feedback configurations. Moreover, nonsymmetry of the effect of one loop on another, a fact not captured by the linear BRG, is accurately predicted by NBRG.

V. Manousiouthakis
M. Nikolaou
Department of Chemical Engineering
University of California
Los Angeles, CA 90024

Introduction

The problem of control structure selection pertains to the configuration of feedback loops that interconnect the input and output control variables already decided upon to use for plant control. Plant decomposition has been indicated to be an appropriate control system design strategy (Umeda et al., 1978; Morari et al., 1980; Morari and Stephanopoulos, 1980), but direct decomposition of the input-output sets (Bristol, 1966; Manousiouthakis et al., 1986) has been recently shown to be more fruitful (Manousiouthakis and McAvoy, 1986).

To evaluate the suitability of such a decomposition, one must assess the interactions among various subsystems. In particular, a quantitative measure of the interactions among feedback loops of different subsystems is necessary. The main interaction measures have been the relative gain array (RGA) (Bristol, 1966), and the Nyquist array methods (Rosenbrock, 1974) and related Gershgorin and Ostrowski bands. Other dynamic interaction measures have also been proposed (Witcher and McAvoy, 1977; Tung and Edgar, 1981; Gagnepain and Seborg, 1982, Grosdidier and Morari, 1986; Manousiouthakis and McAvoy, 1986).

In all cases, the plant was described as a linear, time-invariant system. In various situations, however, such as that for large input signals, this simplifying assumption is inadequate and a nonlinear plant description becomes necessary. It is therefore evident that a measure is needed that assesses the interactions among decentralized feedback loops in the presence of system nonlinearities. This measure must be able to predict interactions when the plant is perturbed far from an operational steady state.

The rest of this paper is structured as follows: The next section presents the mathematical framework, followed by a discussion on the importance of the Block Relative Gain in the linear time-invariant case. Next, the dynamic nonlinear block relative gain (DNBRG) and the nonlinear block relative gain (NBRG) are introduced and their properties are discussed. Finally, an illustrative example is presented and conclusions are drawn.

Mathematical Framework

The input-output system formulation (familiar to process control engineers in the linear time-invariant case as the transfer function approach) has been extended to nonlinear systems in Zames (1966a, b), Willems (1971), and Desoer and Vidyasagar (1975). Elements of this approach are outlined next so that the reader can follow the developments in the sequel.

Correspondence concerning this paper should be addressed to V. Manousiouthakis.

Input-output theory views signals as real, vector-valued functions, and systems as mappings of input signals to output signals. Physical characteristics of signals (e.g., bounded magnitude and finite energy) are described by well defined mathematical properties of the related functions.

In this context the magnitude of a signal x: $R \supseteq S \rightarrow R^n$ is quantified through its p-norm ($p \in [1, \infty]$) defined as follows:

$$\|x\|_p \triangleq \begin{cases} \sup_{t \in S} \|x(t)\|, & p = \infty \\ \left[\int_S \|x(t)\|^p \, dt\right]^{1/p}, & p \in [1, \infty) \end{cases}$$

where $\|x(t)\|$ denotes any norm of the real vector $x(t)$. Typically $S = [0, \infty)$ for piecewise continuous functions of time.

Signals with finite p-norms form a (*Banach*) space L_p^n, defined as follows:

$$L_p^n \triangleq \{x: [0, \infty) \rightarrow R^n, \|x\|_p < \infty\}$$

For example, in the scalar case a signal belongs to L_∞^1 if its value is bounded for all $t \geq 0$.

Physical signals, however large, are always piecewise continuous and finite. Therefore, for a physical signal x and a finite time interval S, $\|x\|_p$ is always finite. However, if $S = [0, \infty)$, then $\|x\|_\infty$, $p = \infty$, if and only if $\|x(t)\|$ grows with no bound. Thus if one assigns to the *physical* time function x: $[0, \infty) \rightarrow R^n$ a *truncated* signal

$$x_T: t \rightarrow x_T(t) \triangleq \begin{cases} x(t) & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases}$$

then $\|x_T\|_p$ will be finite. This naturally leads to the definition of an *extended* (Banach) space L_{pe}^n defined as

$$L_{pe}^n \triangleq \{x: [0, \infty) \rightarrow R^n: x_T \in L_p^n \text{ for all } T \geq 0\}$$

Within this setting the dynamic behavior of a nonlinear system can be described by an *unbiased* nonlinear operator.

$$N: L_{pe}^m \rightarrow L_{pe}^n: u \rightarrow y: 0 \rightarrow N(0) = 0$$

which processes input signals u to output signals y. N is commonly realized through a set of differential (*dynamics*) and algebraic (*output*) equations.

The (*dynamic*) *p-gain* of such a system is defined as

$$\|N\|_p \triangleq \sup_{u,T} \{R(u, T) \triangleq \|Nu_T\|_p / \|u_T\|_p: u \in L_{pe}^n - \{0\}, T \geq 0\}$$

Notice that $\|Nu_T\|_p$ and $\|u_T\|_p$ are always finite numbers.

It is clear that $T_1 > T_2$ implies $\sup_{u \neq 0} \{R(u, T_1)\} > \sup_{u \neq 0} \{R(u, T_2)\}$ because the search for the left hand side supremum is performed over a wider range of input signals u, namely those that are identically zero after $t = T_1$, as compared to those that identically vanish after $t = T_2 < T_1$. Therefore if one restricts the domain of N to signals in L_p^n it holds that

$$\|N\|_p \triangleq \lim_{T \rightarrow \infty} \sup_u \{R(u, T) \triangleq \|Nu_T\|_p / \|u_T\|_p: u \in L_p^n - \{0\}, T \geq 0\}$$

The operator N is defined to be *stable* if its p-gain is finite. According to the preceding discussion a stable operator maps every input signal u in L_p^n to an output signal $y \triangleq Nu$ that also belongs to L_p^n. This is in agreement with the familiar notion of BIBO stability of linear time-invariant systems.

To quantify a nonlinear system's *static* behavior one can concentrate on the extended space $L_{\infty e}^m$ of all signals with *finite* values, and restrict the domain of N to the subspace $SL_{\infty e}^m$ of $L_{\infty e}^m$ that consists of *finite* step functions. Given N, the corresponding operator N_R: $L_{\infty e}^m \rightarrow L_{\infty e}^n$ assigns to each input step function a constant-valued output function y_R, namely

$$N_R: u_R \rightarrow y_R$$

where

$$u_R: t \rightarrow u_R(t) \triangleq \begin{cases} u_o \in R^m & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$y_R: t \rightarrow y_R(t) \triangleq \begin{cases} \lim_{\sigma \rightarrow \infty} (Nu_R)(\sigma) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Therefore, if the asymptotic response of N to a set of bounded step signals is constant, a *function*

$$g: R^m \rightarrow R^n: 0 \rightarrow 0: u_o \rightarrow \lim_{\sigma \rightarrow \infty} (Nu_R)(\sigma)$$

can be used to describe the static behavior of N, and the *static* ∞*-gain* of the nonlinear system is

$$\|N_R\|_\infty = \sup_u \{\|g(u)\| / \|u\|, u \in U \subseteq R^m - \{0\}\} \quad (1)$$

In the special case $n = 1$ (single input), a static p-gain of the system can be introduced as

$$\|g\|_p = \left[\int_U \|g(u)\|^p \, du\right]^{1/p} \quad (2)$$

where the integration domain U explicitly quantifies the magnitude of the input step signals.

Importance of the Linear *BRG*

Aside from the amount of interaction between decentralized feedback loops, the linear *BRG* has been found to provide more information on the closed-loop system characteristics.

Let $G(s)$ be a plant's transfer matrix, $\kappa_2(G)$ the Euclidean condition number of $G(j\omega)$ and $\kappa_2^*(G)$ the condition number of an optimally scaled version of G, i.e.

$$\kappa_2^*(G) \triangleq \inf \{\kappa_2(D_1 G D_2): D_1, D_2 \text{ real, diagonal and positive}\}$$

If $\Lambda(G)$ is the $RGA(j\omega)$ corresponding to G and $\rho(BRG)$ the spectral radius of a $BRG(j\omega)$, then the following two inequalities hold (Nett and Manousiouthakis, 1987):

$$\rho(BRG) \leq \frac{(\kappa_2^*(G) + 1)^2}{4\kappa_2^*(G)}$$

$$2\max \{\|\Lambda(G)\|_1, \|\Lambda(G)\|_\infty\} \leq \kappa_2^*(G) + \frac{1}{\kappa_2^*(G)}$$

Since it is known that the closed-loop robustness of plants with high condition numbers is poor, the above inequalities suggest that a plant with large *BRG*'s imposes serious restrictions on the robustness characteristics of the corresponding closed loop.

In addition to the above, a closed-loop system's reliability against sensor or actuator failures has been shown to be related to its corresponding *RGA* in the following way (Grosdidier and Morari, 1985): If a completely decentralized controller $(k/s)C(s)$ is used to control a plant $G(s)$, then, for a closed-loop stable system with $G(s)C(s)$ proper and the individual decentralized loop between y_i and u_j stable, the breaking of the ij-loop causes instability if $RG_{ij}(0) < 0$.

Finally, limitations on the achievable decentralized closed-loop performance of a system are posed by the *BRG*. It has been shown (Arkun and Manousiouthakis, 1986) that for two-block decentralized feedback control with 1-1/2-2 pairings it holds

$$\left[\frac{\partial y_2(s)}{\partial y_1^{sp}(s)}\right]\cdot\left[\frac{\partial y_1(s)}{\partial y_1^{sp}(s)}\right]^{-1} = -[I + (BRG_{22}(s))^{-1}G_{22}(s)K_2(s)]^{-1}G_{21}(s)G_{11}^{-1}(s) \triangleq S_{21}$$

and

$$\left[\frac{\partial y_1(s)}{\partial y_2^{sp}(s)}\right]\cdot\left[\frac{\partial y_2(s)}{\partial y_2^{sp}(s)}\right]^{-1} = -[I + (BRG_{11}(s))^{-1}G_{11}(s)K_1(s)]^{-1}G_{12}(s)G_{22}^{-1}(s) \triangleq S_{12}$$

The quantities S_{12} and S_{21} should be small for the closed-loop system to exhibit small interactions. If y_1 is a scalar

$$S_{21}S_{12} = [I + (BRG_{22}(s))^{-1}G_{22}(s)K_2(s)]^{-1} \cdot [1 - (BRG_{11}(s))^{-1}]\,[1 + (BRG_{11}(s))^{-1}G_{11}(s)K_1(s)]^{-1}$$

The above equation indicates that the product of the relative sensitivities S_{12} and S_{21} depends only on the Relative Gains, the controllers K_1, K_2 and the transfer functions G_{11} and G_{22}. The factors G_{12} and G_{21} do not appear explicitly, although they affect $S_{12}S_{21}$ through the Relative Gains. The magnitude of $S_{12}S_{21}$ is directly proportional to the magnitude of the factor $(1 - BRG_{11}^{-1})$ which does *not* depend on the controllers K_1, K_2. Therefore, if $(1 - BRG_{11}^{-1})$ is large at the open loop system's crossover frequency, then significant closed-loop interactions should be expected.

Nonlinear Block Relative Gain

In the spirit of Manousiouthakis et al. (1986), we consider a plant N with m of its n inputs interconnected to m of its n outputs through a single (in general MIMO) feedback controller K_1. The remaining $n - m$ inputs and outputs can also be partitioned in respective groups, interconnected through feedback controllers $K_2, K_3, \ldots, K_r$. We shall incorporate all these controllers into a single block, named K_2. The resulting decentralized feedback configuration is depicted in Figure 1. The following relations then hold:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \triangleq y = Nu \quad (\text{i.e., } y(t) = (Nu)(t) \quad t \geq 0) \tag{3}$$

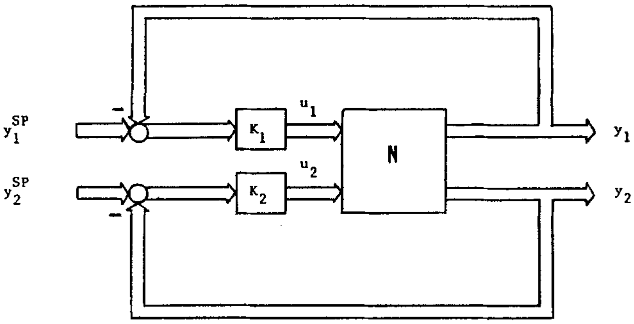


Figure 1. Decentralized feedback.

whence

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \triangleq u = N^{-1}y \text{ (assuming } N^{-1} \text{ exists)} \tag{4}$$

$$u = Ke \tag{5}$$

$$e = y^{sp} - y \tag{6}$$

In order to assess how much the first loop is affected by the remaining loops (designated as loop 2), consider the following two cases:

(i) *Loop 1 closed, loop 2 open (feedback of the specified* m *outputs to the corresponding* m *inputs).* Then assuming that all variables are measured with respect to their values at a steady state and all inverses of operators considered are well defined, we have that

$$u_2(t) = 0 \quad t \geq 0$$

Then, by Eq. 3

$$y_1 = N_1(u_1, u_2 = 0) \triangleq N_{11}u_1 \longrightarrow u_1 = N_{11}^{-1}y_1$$

and, by Eq. 5,

$$u_1 = K_1e_1 \longrightarrow e_1 = K_1^{-1}u_1 = K_1^{-1}(N_{11}^{-1}y_1) = (N_{11}K_1)^{-1}y_1$$
$$(6) \longrightarrow = y_1^{sp} - y_1 = e_1\,(N_{11}K_1)^{-1}y_1 \longrightarrow [I + (N_{11}K_1)^{-1}]y_1 = y_1^{sp} \longrightarrow y_1 = [I + (N_{11}K_1)^{-1}]^{-1}y_1^{sp} \tag{7}$$

(ii) *Loop 1 closed, loop 2 closed (full feedback).* We will determine again the response between y_1 and y_1^{sp}.

Assuming that $y_2^{sp} = 0$ we get from Eq. 6

$$e_2 = -y_2$$

Equations 5 and 3 imply:

$$u_2 = K_2e_2 \longrightarrow e_2 = K_2^{-1}u_2$$
$$y_2 = N_2(u_1, u_2)$$

From the last three relations we obtain

$$N_2(u_1, u_2) + K_2^{-1}u_2 = 0 \longrightarrow u_2 = \Phi_1u_1 \tag{8}$$

Substituting into Eq. 3 we get

$$y_1 = N_1(u_1, \Phi_1 u_1) \hat{=} N'_{11} u_1$$

By arguments similar to those of case (i) we finally obtain

$$y_1 = [I + (N'_{11}K_1)^{-1}]^{-1} y_1^{sp}$$

which can be rewritten as

$$\begin{aligned} y_1 &= [I + (N'_{11}N_{11}^{-1}N_{11}K_1)^{-1}]^{-1} y_1^{sp} \\ &= [I + (N_{11}N_{11}^{-1}N'_{11}K_1)^{-1}]^{-1} y_1^{sp} \\ &= [I + ((DNBRG_l)^{-1}N_{11}K_1)^{-1}]^{-1} y_1^{sp} \\ &= [I + (N_{11}(DNBRG_r)^{-1}K_1]^{-1} y_1^{sp} \end{aligned} \quad (9)$$

where we define $DNBRG_l$ and $DNBRG_r$ as the operators:

$$DNBRG_l = N_{11}(N'_{11})^{-1}: L_{pe}^m \rightarrow L_{pe}^m \quad (10)$$

and

$$DNBRG_r = (N'_{11})^{-1}N_{11}: L_{pe}^m \rightarrow L_{pe}^m \quad (11)$$

Remarks

• Equation 9 indicates that the closed-loop performance of the $m \times m$ loop under consideration is a function of *DNBRG*, as dictated thereby. In fact, if one wanted to design the controller K_1 in a decentralized fashion, then one should perform the design not for N_{11} but for $(DNBRG_l)^{-1}N_{11} = N_{11}(DNBRG_r)^{-1} = N'_{11}$.

• It should be stressed that $DNBRG_l$ and $DNBRG_r$ depend on the controller K_2 as Eqs. 8 and 9 imply. Thus the design of K_1 and K_2 should be carried out in a simultaneous fashion since the one affects the other.

• It is obvious by Eq. 9 that for $DNBRG_{l,r} = I$ the operation of the feedback loops that correspond to the remaining $(n - m)$ inputs and outputs does not affect the performance of the first $m \times m$ loop. In such case the controller K_1 could be designed first, without account to the controller K_2. Therefore, in order to quantify the effect of loop 2 on loop 1, the distance of the operator $DNBRG_{l,r}$ from the identity operator in some topological sense must be assessed.

Although *DNBRG* is derived without any simplifying assumptions, it is difficult to use for design purposes since it depends on the controller K that is to be designed at a subsequent step, i.e., after the selection of the decentralized feedback configuration. Hence, we make the assumption that the plant output y_2 is under perfect control, i.e.

$$y_2(t) = 0 \quad t \geq 0 \quad (12)$$

Then, in analogy to the previous paragraph, we have:

(i) Loop 1 closed, loop 2 open: In this case there is no change and Eq. 7 holds.

(ii) Loop 1 closed, loop 2 closed: Then, we have by Eq. 4:

$$u_1 = (N^{-1})_1 (y_1, y_2 = 0) \hat{=} (N^{-1})_{11} y_1 \Longrightarrow y_1 = (N^{-1})_{11}^{-1} u_1$$

and, in analogy to Eq. 7 we get

$$\begin{aligned} y_1 &= [I + ((N^{-1})_{11}^{-1}K_1)^{-1}]^{-1} y_1^{sp} \\ &= [I + ((N_{11}(N^{-1})_{11})^{-1}N_{11}K_1)^{-1}]^{-1} y_1^{sp} \\ &= [I + (N_{11}((N^{-1})_{11}N_{11})^{-1}K_1)^{-1}]^{-1} y_1^{sp} \end{aligned}$$

From the above equation we define $\overline{DNBRG}_l$ and $\overline{DNBRG}_r$, respectively, as the operators:

$$\begin{aligned} \overline{DNBRG}_l &= N_{11}(N^{-1})_{11}: L_{pe}^m \rightarrow L_{pe}^m \\ \overline{DNBRG}_r &= (N^{-1})_{11}N_{11}: L_{pe}^m \rightarrow L_{pe}^m \end{aligned} \quad (13)$$

Remarks

• For plants described by the linear input-output equation $y(s) = G(s)u(s)$ the above two definitions are analogous to the classical *BRG* introduced by Manousiouthakis et al. (1986).

• Contrary to *DNBRG*, $\overline{DNBRG}$ does *not* depend on the controller and assesses the closed-loop performance using only plant information.

• Input (output) scaling does not affect $\overline{DNBRG}_l$ ($\overline{DNBRG}_r$) while output (input) scaling does, as shown next. Let

$$\begin{aligned} y_1 &= N_{11}u_1 \\ u_1 &= (N^{-1})_{11}y_1 \end{aligned}$$

Under the scaling $y_i = Y_i y'_i$ and $u_j = U_j u'_j$ the above equations yield

$$\begin{aligned} y'_1 &= Y_1^{-1}(N_{11}U_1)u'_1 \hat{=} \tilde{N}_{11}u'_1 \\ u'_1 &= U_1^{-1}((N^{-1})_{11}Y_1)y'_1 \hat{=} (\tilde{N}^{-1})_{11}y_1 \end{aligned}$$

Thus

$$\begin{aligned} \overline{DNBRG}'_l = \tilde{N}_{11} \cdot (\tilde{N}^{-1})_{11} &= Y_1^{-1}(N_{11}U_1)[U_1^{-1}((N^{-1})_{11}Y_1)] \\ &= Y_1^{-1}N_{11}U_1U_1^{-1}(N^{-1})_{11}Y_1 = Y_1^{-1}N_{11}(N^{-1})_{11}Y_1 \end{aligned}$$

and similarly,

$$\overline{DNBRG}'_r = U_1^{-1}(N^{-1})_{11}N_{11}U_1$$

The last two equations suggest that even for completely decentralized control, input and output scaling is significant. Use of dimensionless output variables is therefore indicated.

• Evaluation of the distance of $\overline{DNBRG}$ from the identity operator can be performed by using the procedure outlined in Nikolaou and Manousiouthakis (1988). In particular, the expression to optimize is:

$$\sup_{u \in U} \frac{\|(\overline{DNBRG} - I)u\|_p}{\|u\|_p}$$

where the search set U of physically meaningful input signals u must be carefully chosen. It is evident that $\overline{DNBRG} - I$ may not have, in the above sense, a finite gain on U. This will be an indication of a severe effect of the remaining feedback loops, on the loop between u_1 and y_1.

• It is a common conjecture in control theory (Freudenberg and Looze, 1985) that multivariable plants with large condition number are difficult to control. In support of the conjecture that multivariable plants with large $\overline{NBRG}$ are inherently difficult to control, we show next that $\|\overline{NBRG}\|$ constitutes a lower bound for the condition number of a nonlinear system $N: L_{pe}^n \rightarrow L_{pe}^n$. To establish this, observe that $N_{11} = P_1NP_2$ and $N_{11}^{-1} = P_2N^{-1}P_1$ where P_1 and P_2 are projection operators, i.e. map an n-component vector to one with $n - m$ components equal to 0 ($m \leq n$). Then $\|\overline{NBRG_l}\| = \|N_{11}(N^{-1})_{11}\| = \|P_1 N P_2 P_2 N^{-1} P_1\| \leq \|P_1\| \cdot \|N\| \cdot \|P_2\| \cdot \|P_2\| \cdot \|N^{-1}\| \cdot \|P_1\|$. But the norm of any projection operator is 1, therefore

$$\|\overline{NBRG_l}\| \leq \|N\| \cdot \|N^{-1}\|$$

The righthand side of the above inequality constitutes the condition number ϵ of N.

To avoid the complexity originating from the need to compute the distance of $\overline{DNBRG}$ from the identity operator I and based on the relation between the linear BRG and relative sensitivities, demonstrated in Arkun and Manousiouthakis (1986), we focus on the steady-state behavior of the plant, which is described by equations analogous to Eqs. 3–6:

$$y = g(u), \ (g: \boldsymbol{R}^n \rightarrow \boldsymbol{R}^n)$$
$$u = g^{-1}(y), \text{ (assuming } g^{-1} \text{ exists)}$$
$$u = K(e)$$
$$e = y^{sp} - y$$

Then we define the $(NBRG_l)_{ij}$ and $(NBRG_r)_{ij}$ that correspond to the input vector u_j and the output vector y_i as

$(NBRG_l)_{ij} = g_{ij} \cdot (g^{-1})_{ji}: \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m$:

$$y_i \xrightarrow[y_{k \neq 1}=0]{} u_j \xrightarrow[u_{k \neq j}=0]{} \bar{y}_i \tag{14}$$

$(NBRG_r)_{ij} = (g^{-1})_{ji} \cdot g_{ij}: \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m$:

$$u_j \xrightarrow[u_{k \neq j}=0]{} y_i \xrightarrow[y_{k \neq i}=0]{} \bar{u}_j \tag{15}$$

Remarks

- As Eqs. 14 and 15 indicate *NBRG* is a vector-valued function (resulting from the composition of two functions) whose distance from the identity function can be computed by virtue of Eqs. 1 and 2.
- For linear, time-invariant systems definitions (Eqs. 14 and 15) reduce to the familiar *BRG*. In particular, if the system stays close to the operating steady state, *NBRG* can be arbitrarily close to *BRG*.

Table 1. Functions Required for the Evaluation of NBRG for CSTR

$$(g_{11})^{-1}: u_1 = \frac{\rho c_p F_s}{Q_s}\left[(T_i - T_s(1 + y_1)) + \frac{Jk_o \exp\left[-\dfrac{E}{RT_s(1 + y_1)}\right] C_{Ai}}{\dfrac{F_s}{V} + k_o \exp\left[-\dfrac{E}{RT_s(1 + y_1)}\right]}\right] - 1$$

$$(g^{-1})_{11}: u_1 = \frac{\rho c_p V k_o}{Q_s} \exp\left[-\frac{E}{RT_s(1 + y_1)}\right] C_{As}\left[\frac{T_i - T_s(1 + y_1)}{C_{Ai} - C_{As}} + J\right] - 1$$

$$(g_{12})^{-1}: u_2 = \frac{Q_s}{F_s \rho c_p (T_i - T_s(1 + y_1))} - Jk_o \exp\left[-\frac{E}{RT_s(1 + y_1)}\right] \frac{1}{T_i - T_s(1 + y_1)} \cdot \frac{(1 + u_2) C_{Ai}}{\dfrac{F_s(1 + u_2)}{V} + k_o \exp\left[-\dfrac{E}{RT_s(1 + y_1)}\right]} - 1$$

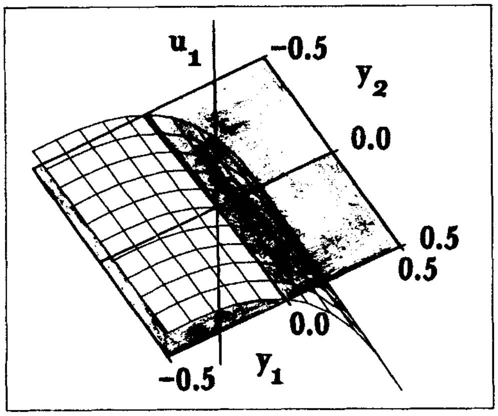
$$(g^{-1})_{21}: u_2 = \frac{k_o \exp\left[-\dfrac{E}{RT_s(1 + y_2)}\right]}{F_s} \cdot \frac{C_{As} V}{C_{Ai} - C_{As}} - 1$$

$$(g_{21})^{-1}: u_1 = \frac{\rho c_p F_s}{Q_s}\left[T_i + J(C_{Ai} - C_{As}(1 + y_2) + \frac{E/R}{\ln\left[\dfrac{F_s(C_{Ai} - C_{As}(1 + y_2))}{V k_o C_{As}(1 + y_2)}\right]}\right] - 1$$

$$(g^{-1})_{12}: u_1 = \frac{\rho c_p V}{Q_s}\left[\frac{k_o \exp\left[-\dfrac{E}{RT_s}\right] C_{As}(1 + y_2)}{C_{Ai} - C_{As}(1 + y_2)}(T_i - T_s) + Jk_o \exp\left[-\frac{E}{RT_s}\right] C_{As}(1 + y_2)\right] - 1$$

$$(g_{22})^{-1}: u_2 = \left[\frac{Q}{\rho c_p F_s}\right] \Bigg/ \left[T_i + \frac{E/R}{\ln\left[\dfrac{F_s(1 + u_2)(C_{Ai} - C_{As}(1 + y_2))}{V k_o C_{As}(1 + y_2)}\right]} + J(C_{Ai} - C_{As}(1 + y_2))\right] - 1$$

$$(g^{-1})_{22}: u_2 = \frac{V k_o \exp\left[-\dfrac{E}{RT_s}\right] C_{As}(1 + y_2)}{F_s(C_{Ai} - C_{As}(1 + y_2))} - 1$$



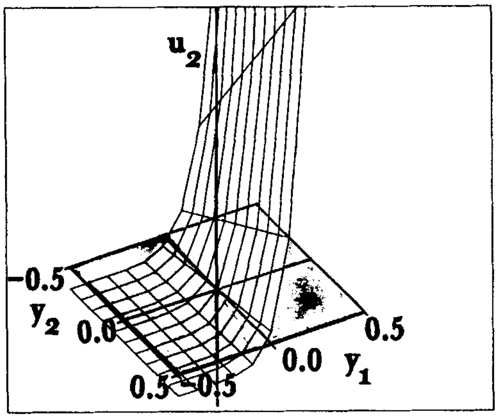


Figure 2a, b. u_1 and u_2 as functions of (y_1, y_2).

• The functions g_{ij} and $(g^{-1})_{ji}$ are easy to obtain experimentally by measuring the steady-state response of y_i to step changes of various magnitudes in u_j, when:

(i) No feedback loop is closed (to obtain g_{ij}).

(ii) There is no feedback between y_i and u_j while all other outputs and inputs are interconnected through feedback loops (to obtain $(g^{-1})_{ji}$).

• Since for a physical system arbitrarily large deviations from a steady-state point are not meaningful, the function g is not defined on the entire R^n but on a (closed and bounded) subset of R^n. It is therefore a matter to search, for what subsets of R^n, containing a certain steady state, g^{-1} exists and is unique. In particular, if

$$A \hat{=} \left[\frac{\partial g}{\partial x}\right]_{x=0}$$

then the inverse function theorem suggests that g^{-1} exists in a neighborhood of 0 if A^{-1} exists or, equivalently, if

$$\det A \neq 0$$

Given that for the linear case,

$$BRG_{ij} = A_{ij}(A^{-1})_{ji}$$

we conclude that existence of *BRG* implies the existence of *NBRG* in a neighborhood of 0. The range of that neighborhood, however, is not obvious.

Table 2. CSTR Parameter Values

V	1.359 m^3	$-\Delta H_R$	69,775 J/mol
T_i	372.2 K	$J = \Delta H_R/\rho C_p$	0.02774 K · m^3/mol
C_{Ai}	8,008.5 mol/m^3	Steady State	
k_o	7.08×10^{10} h^{-1}	F_s	2.832 m^3/h
E/R	8,333.3 K	Q_s	9.891×10^7 J/h
ρ	800.8 kg/m^3	T_s	580.5 K
c_p	3,140 J/kg · K	C_{As}	0.4084 mol/m^3

• *NBRG* can be used as an analysis tool by screening out those feedback configurations for which *NBRG* is far from the identity operator. To reduce the number of alternatives to be considered, the linear *BRG* can be used first as a synthesis tool to indicate promising pairings between inputs and outputs.

Example

We shall study a CSTR modeled by the equations (Stephanopoulos, 1984)

$$\frac{dy_1}{dt} = \frac{F_s(1+u_2)}{V}\left(\frac{T_i}{T_s} - 1 - y_1\right) + \frac{Jk_o}{T_s}\exp\left[-\frac{E}{RT_s(1+y_1)}\right]C_{As}(1+y_2) - \frac{Q_s(1+u_1)}{\rho c_p V T_s} \quad (16)$$

$$\frac{dy_2}{dt} = \frac{F_s(1+u_2)}{V}\left(\frac{C_{Ai}}{C_{As}} - 1 - y_2\right) - k_o \exp\left[-\frac{E}{RT_s(1+y_1)}\right](1+y_2) \quad (17)$$

where $J = (-\Delta H_R)/\rho C_p$, $y_1 = (T - T_s)/T_s$, $y_2 = (C_A - C_{As})/C_{As}$, $u_1 = (Q - Q_s)/Q_s$, $u_2 = (F - F_s)/F_s$. At steady state Eqs. 16 and 17 define a function

$$g: R^2 \longrightarrow R^2,\ u \longrightarrow y = g(u)$$

Consider now the function

$$g_{11}: u_1 \longrightarrow y_1 \text{ when } u_2 = 0$$

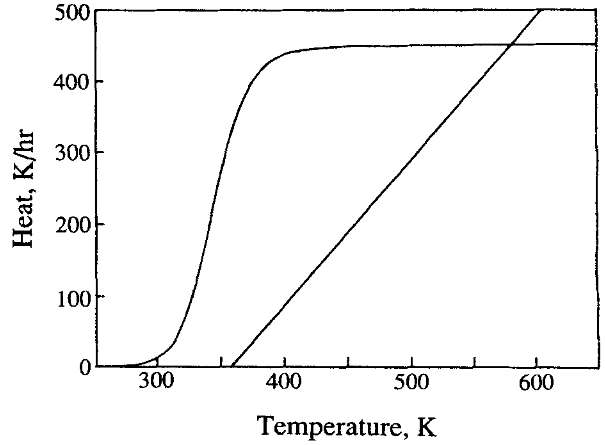


Figure 3. Evaluation of steady state of CSTR.

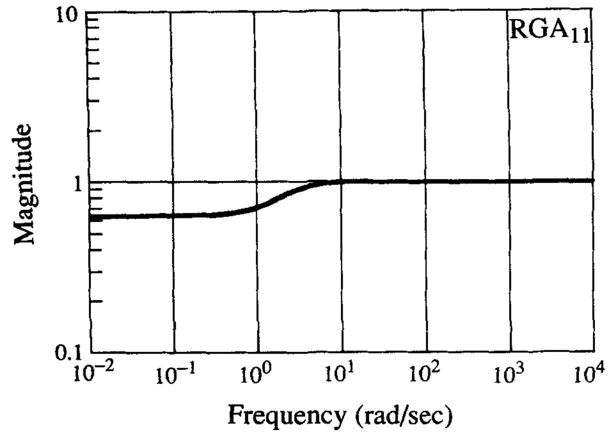


Figure 4. Dynamic linear RG_{11} and RG_{22} for CSTR.

To define g_{11} solve Eq. 17 for y_2 at steady state and substitute into Eq. 16 to get

$$(g_{11})^{-1}: u_1 = \frac{\rho c_p F_s}{Q_s} \cdot \left[(T_i - T_s(1+y_1)) + \frac{J k_o \exp\left[-\frac{E}{RT_s(1+y_1)}\right] C_{Ai}}{\frac{F_s}{V} + k_o \exp\left[-\frac{E}{RT_s(1+y_1)}\right]} \right] - 1$$

Next, to find

$$(g^{-1})_{11}: u_1 \rightarrow y_1 \text{ when } y_2 = 0$$

solve Eq. 17 for u_2 at steady state and substitute into Eq. 16 to get

$$(g^{-1})_{11}: u_1 = \frac{\rho c_p V k_o}{Q_s} \cdot \exp\left[-\frac{E}{RT_s(1+y_1)}\right] C_{As} \left[\frac{T_i - T_s(1+y_1)}{C_{Ai} - C_{As}} + J\right] - 1$$

Similarly we can get the remaining functions that are required. (Explicit formulae are shown in Table 1). To check the invertibility of g, 3-D plots of u_1 and u_2 as functions of (y_1, y_2) are constructed (Figure 2). As can be seen, within the range of a, b considered below invertibility is guaranteed. For the parameter values presented in Table 2 we shall calculate numerically the integrals

$$I_{ij} = \left[\int_a^b \{[g_{ij} \cdot (g^{-1})_{ji} - I](x)\}^2 \, dx\right]^{1/2} \quad i, j = 1, 2$$

where

$$a < 0 \text{ and } b > 0$$

We shall compare our method to the linear approach.

A single stable steady state for the CSTR exists (Figure 3). For the linearized model, RGA is

$$RGA = \begin{bmatrix} 0.657 & 0.343 \\ 0.343 & 0.657 \end{bmatrix}$$

indicating that a configuration 1-1/2-2 should be preferred. In addition, the dynamic linear RG's are presented in Figures 4 and 5; they strongly support the pairing 1-1/2-2. SISO controllers were designed using the Model Reference Scheme for each element of the transfer matrix $G(s)$ (Table 3). First order filters with time constant 0.1 hr were used. The resulting controllers are shown in Table 4.

For the nonlinear model the distances of $NBRG$'s from the identity operator were computed, according to Eq. 2, for $a = -0.35$ to $a = -0.05$ and $b = 0.05$ to $b = 0.45$. The results are presented in Figures 6 through 9. Numerical simulations of the closed-loop system for two different feedback configurations are presented in Figures 10 through 13.

Remarks

- As Figures 10–13 indicate, the feedback configuration 1-1/2-2 is preferable to 1-2/2-1, although they both exhibit significant interactions. In addition to the linear approach, which indicates that the 1-1/2-2 loops interact less than the 1-2/2-1

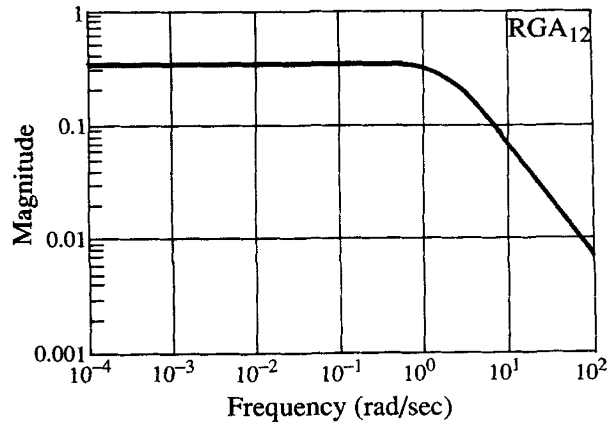


Figure 5. Dynamic linear RG_{12} and RG_{21} for CSTR.

Table 3. Transfer Matrix between (u_1,u_2) and (y_1,y_2) for Linearized Model of CSTR

$$G(s) = \frac{1}{s^2 + 40{,}844s + 85{,}087} \cdot \begin{bmatrix} -0.049840s - 2{,}036.2 & -0.74761s + 2{,}032.9 \\ 29{,}226 & 40{,}848s + 55{,}897 \end{bmatrix}$$

Table 4. Decentralized CSTR Controllers for Two Different Feedback Configurations

Pairing 1-1/2-2: $K_1(s) = -\dfrac{n(s)}{(0.04984s + 2{,}036.2)0.1s}$

$K_2(s) = \dfrac{n(s)}{(40{,}848s + 55{,}897)0.1s}$

Pairing 1-2/2-1: $K_1(s) = \dfrac{n(s)}{(0.74761s + 2{,}047.8)0.1s}$

$K_2(s) = \dfrac{n(s)}{29{,}226(0.1s + 2)0.1s}$

$n(s) = s^2 + 40{,}844s + 85{,}087$

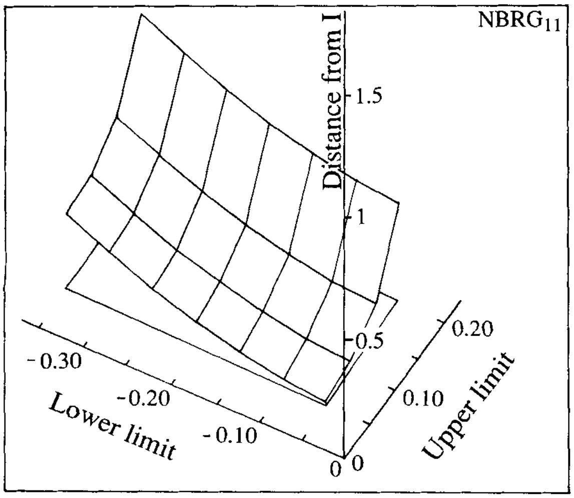


Figure 6. Distance of $NBRG_{11}$ from I (vertical axis) as a function of a and b (horizontal axes).

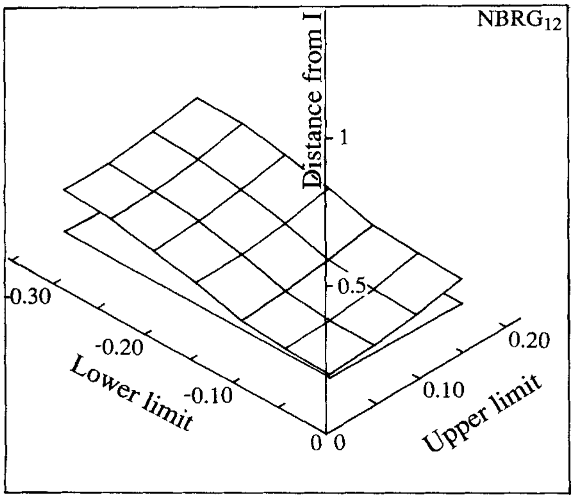


Figure 7. Distance of $NBRG_{12}$ from I (vertical axis) as a function of a and b (horizontal axes).

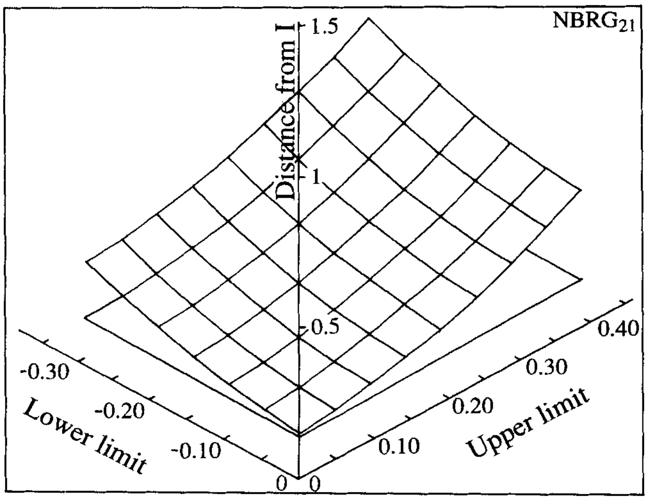


Figure 8. Distance of $NBRG_{21}$ from I (vertical axis) as a function of a and b (horizontal axes).

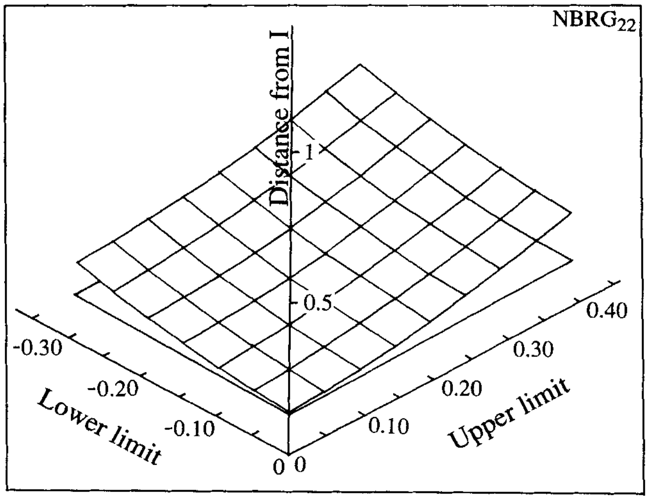


Figure 9. Distance of $NBRG_{22}$ from I (vertical axis) as a function of a and b (horizontal axes).

loops, the nonlinear approach provides more information: In configuration 1-2/2-1 $NBRG_{11}$ is in general farther from the identity than is $NBRG_{22}$. This suggests that nonlinearity has a stronger effect on the output responses to y_1^{sp} changes, rather than to y_2^{sp} changes. This is exemplified in Figures 12 and 13 where for y_1^{sp} changes of different magnitudes, y_1 and y_2 deviate significantly from the linearized system's response (essentially obtained for $y_1^{sp} = 0.02$), while the responses of y_1, y_2 to various y_2^{sp} changes virtually coincide. This behavior could not have been predicted by the linear BRG, for which $BRG_{ij} = BRG_{ji}$. It should be stressed that even if the linear BRG was considered for the linearized model around different points ranging from the initial to the terminal steady state, no prediction of the asymmetrical influence of one loop on the other could have been made, due to the fact that the RGA is always symmetric.

- The limiting behavior of $NBRG$ when input signals are small, is close to BRG. In particular, BRG is the derivative of $NBRG(x)$ with respect to x, at $x = 0$. Therefore it should be

$$\lim_{x \to 0} \frac{NBRG(x) - NBRG(0)}{x - 0} = \lim_{x \to 0} \frac{NBRG(x)}{x} = BRG$$

This is shown in Figures 14 and 15.

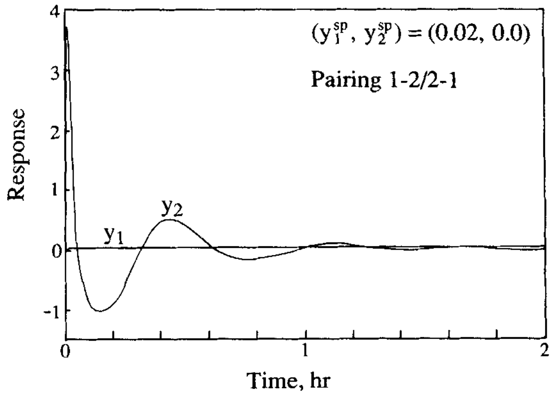


Figure 10. Responses of y_1 and y_2 to step change on y_1^{sp} for feedback pairing 1-2/2-1.

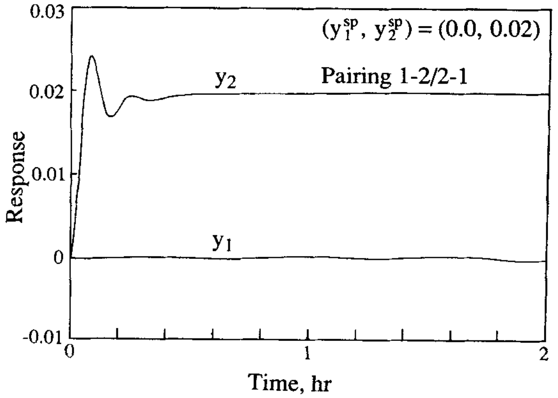


Figure 11. Responses of y_1 and y_2 to step change on y_2^{sp} for feedback pairing 1-2/2-1.

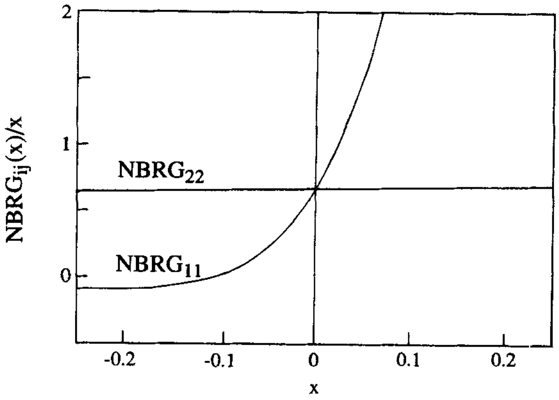


Figure 14. Behavior of $NBRG_{11}(x)/x$ and $NBRG_{22}(x)/x$ as $x \rightarrow 0$.

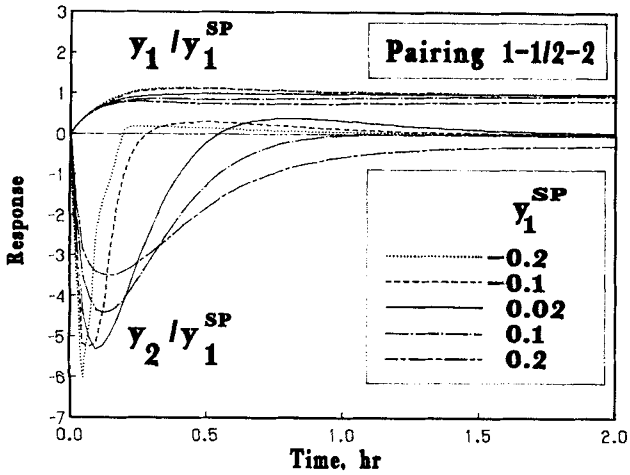


Figure 12. Responses of y_1 and y_2 to step change on y_1^{sp} for feedback pairing 1-1/2-2.

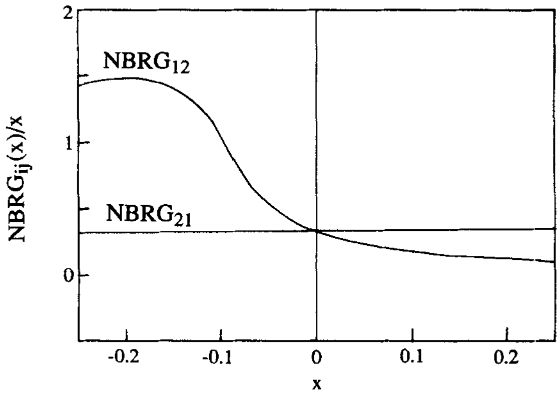


Figure 15. Behavior of $NBRG_{12}(x)/x$ and $NBRG_{21}(x)/x$ as $x \rightarrow 0$.

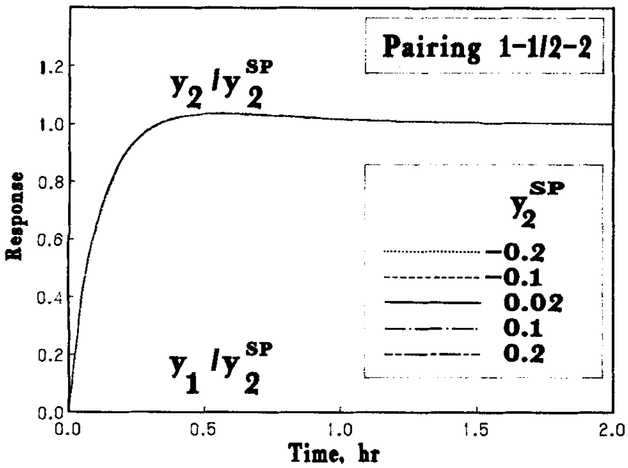


Figure 13. Responses of y_1 and y_2 to step change on y_2^{sp} for feedback pairing 1-1/2-2.

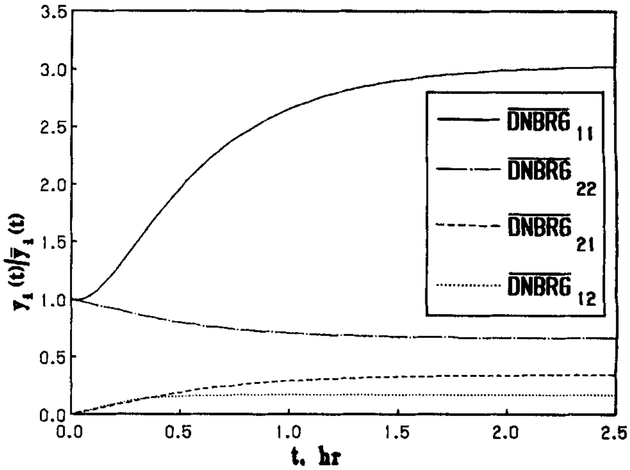


Figure 16. Responses of $(\overline{DNBRG_i})ij$ to $\bar{y}_i(t) = 0.1 (1 - \exp(-10t))$.

• Care must be taken in the inversion of the functions involved. For larger inputs, multiplicities and/or unstable steady states may exist. This is a situation to be investigated.

• One can obtain a realization of $(\overline{DNBRG_l})_{ij} \hat{=} N_{ij}(N^{-1})_{ji}$ as follows:

a. For $y_k = 0$, $k \neq i$, solve Eqs. 16 and 17 algebraically with respect to u_j, to create a realization of $(N^{-1})_{ji}$.

b. For $u_k = 0$, $k \neq j$, and u_j, obtained from the set of equations of step a, set up Eqs. 16 and 17.

One then should perform an optimization to determine

$$\sup_{u \in U} \frac{\| [(\overline{DNBRG_l})_{ij} - I] u \|_p}{\| u \|_p}$$

in order to assess closed-loop interactions under the perfect control assumption (Eq. 12). This goes beyond the scope of this article and is treated in detail in Nikolaou and Manousiouthakis (1988). Instead, we provide in Figure 16 the response $y_i(t) = [(\overline{DNBRG_l})_{ij}\bar{y}_i](t)$, $(i = 1, 2,\ j = 1, 2)$ to inputs $\bar{y}_i(t) = 0.1(1 - \exp(-10t)$. As can be seen the steady-state value of $y_i(t)/\bar{y}_i(t)$ coincides with that of $NBRG_{ij}(x)/x$ in Figures 14 and 15 for $x = 0.1$.

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Notation

BRG_{ij} = block relative gain between y_i and u_j
$\underline{DNBRG}$ = dynamic nonlinear block relative gain
$\overline{DNBRG}$ = restricted $DNBRG$
g = function from R^n to R^n
$G(s)$ = transfer matrix
inf = greatest lower bound of the elements of a set
L_p^n = Banach space of p-integrable functions taking values in R^n
L_{pe}^n = extended Banach space of p-integrable functions taking values in R^n
N = nonlinear operator
$\|N\|_p$ = p norm of the operator N
$NBRG$ = nonlinear block relative gain
R^n = set of real n vectors
RG = relative gain
RGA = relative gain array
sup = least upper bound of the elements of a set
u = manipulated variables
$\|x\|_p$ = p norm of the function $x: R \rightarrow R^n$
y = measured variables
ϵ = condition number of N
$\Lambda(G)$ = $RGA\ (j\omega)$
$\kappa_2(G)$ = Euclidean condition number of the matrix $G(s)$
$\kappa_2^*(G)$ = Euclidean condition number of optimally scaled $G(s)$
$\rho(\cdot)$ = spectral radius

Superscripts and subscripts

l = left
r = right
s = initial steady state
sp = set point

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